

## Solution Derivations for Capa #11

Caution: The symbol  $E$  is used interchangeably for energy and EMF.

1) DATA:  $V_b = 5.0\text{ V}$ ,  $R = 155\ \Omega$ ,  $L = 8.400 \times 10^{-2}\text{ H}$ . In the diagram above, what is the voltage across the inductor in the instant just after the switch is closed?

$V_b = \text{Given}$

$R = \text{Given}$

$L = \text{Given}$

From Kirchhoff's Loop law, we can get an equation for the voltages around the circuit.

$$V_b - IR - E_L = 0$$

where  $E_L$  is the EMF of the inductor. Thus,

$$E_L = V_b - IR$$

Immediately after the switch is closed, there is no current in the circuit. The resistor and inductor are in series, and the inductor opposes the change in current. Thus, there is no voltage drop across the resistor and the voltage drop across the inductor is the initial voltage  $V_b$ .

2) After the switch is closed for a long time, what is the energy stored in the inductor?

The energy stored in an inductor is given by

$$E = \frac{1}{2}LI^2$$

The maximum current in this circuit is determined by the battery and the resistor. From Ohm's law,

$$I = \frac{V}{R}$$

Thus,

$$E = \frac{1}{2}L\left(\frac{V}{R}\right)^2$$

3) The switch in the above diagram is closed after being open a long time. The initial charge on the capacitor is zero. (For each statement select T True, F False).

QUESTION:

A) In the instant after the switch is closed, the voltage across the capacitor equals the voltage across the battery.

B) In the instant after the switch is closed, the voltage across the inductor equals the voltage across the battery.

C) A long time after the switch is closed, the voltage across the capacitor equals

the voltage across the battery.

D) A long time after the switch is closed, the current through the resistor is zero.

ANSWER:

A) False, initially capacitors act like wires without resistance, so there is no voltage drop.

B) True, there is no current elsewhere in the circuit, so there are no other voltage drops. See (#1).

C) True, after a long time a capacitor acts like an infinite resistor, so no current can flow. Thus, the voltage drop must be equal to the battery.

D) True, see (C).

CAPA is looking for an answer in the form FTTT

4) The next 6 questions refer to this situation: An LR circuit is hooked up to a battery as shown in the figure, with the switch initially open. The resistance in the circuit is  $R = 110 \Omega$ , the inductance is  $L = 3.40 H$  and the battery maintains a voltage of  $E = 30.0 V$ . At time  $t = 0$  the switch is closed. What is the current through the circuit after the switch has been closed for  $t = 4.57 \times 10^{-2} s$ ?

$R =$  Given

$L =$  Given

$E =$  Given

$t =$  Given

For a rising current in an RL circuit, the equation for the current at any time is

$$I = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

5) What is the voltage across the inductor after the switch has been closed for  $t = 4.57 \times 10^{-2} s$ ?

This is essentially the same circuit as in #1. Thus, the same equation from Kirchoff's loop law can be applied (or easily derived).

$$E_L = V_b - IR$$

However, there is a current through the resistor at this point which you just calculated.

6) What is the power dissipation in the resistor at  $t = 4.57 \times 10^{-2} s$ ?

The easiest way to calculate the answer is to use the formula for power

$$P = I^2 R$$

Assuming the time is the same for these problems, the current  $I$  was calculated in (#4) and the resistance was given.

7) How much energy is stored in the inductor at  $t = 4.57 \times 10^{-2}$  s?

The energy stored in an inductor is given by

$$U = \frac{1}{2}LI^2$$

where the inductance  $L$  was given and the current  $I$  was calculated previously.

8) How much work has the battery done from the time the switch was closed until  $t = 4.57 \times 10^{-2}$  s?

To solve this problem, you needed to integrate the power equation for the time length specified (i.e. from  $t = 0$  to  $t = t_0$  (Given)). Since this is the work the battery has done, the logical choice for the power equation would be

$$P = IV = IE$$

since the voltage is the EMF. The current is determined by the inductor and changes with time according to the equation

$$I = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

The resulting integral is

$$\begin{aligned} W &= \int_0^{t_0} \left(E \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)\right) dt \\ &= \frac{E^2}{R^2} \left(t_0 R + L e^{-t_0 \frac{R}{L}} - L\right) \end{aligned}$$

I did this on a TI-92, but it can be verified by hand.

9) How much energy has been dissipated in the resistor from the time the switch was closed until  $t = 4.57 \times 10^{-2}$  s?

To find the energy dissipated by the resistor, we must again integrate the power equation of the time interval specified (from  $t = 0$  to  $t = t_0$  (Given)). This time, the best choice of the power equation is

$$P = I^2 R$$

where  $I$  is the current determined by the inductor. It is equal to

$$I = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

Thus, the energy dissipated is

$$\begin{aligned} U &= \int_0^{t_0} R \left(\frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)\right)^2 dt \\ &= \frac{1}{2} \frac{E^2}{R^2} \left(-L + 4L e^{t_0 \frac{R}{L}} + 2t_0 R e^{2t_0 \frac{R}{L}} - 3L e^{2t_0 \frac{R}{L}}\right) e^{-2t_0 \frac{R}{L}} \end{aligned}$$

10) The next three questions refer to this situation. A very long solenoid with a circular cross section and radius  $r_1 = 1.90 \text{ cm}$  with  $n_s = 280 \text{ turns/cm}$  lies inside a short coil of radius  $r_2 = 5.00 \text{ cm}$  and  $N_c = 35 \text{ turns}$ . If the current in the solenoid is ramped at a constant rate from zero to  $I_s = 1.40 \text{ A}$  over a time interval of  $76.0 \text{ ms}$ , what is the magnitude of the emf in the outer coil while the current in the solenoid is changing?

$r_1 = \text{Given}$   
 $n_s = \text{Given}$   
 $r_2 = \text{Given}$   
 $N_c = \text{Given}$   
 $I_s = \text{Given}$   
 $t = \text{Given}$

This problem is easier if you do #11 first. The magnitude of the EMF is given by

$$E_2 = -M \frac{dI_s}{dt}$$

Which we can calculate by

$$E_2 = -M \frac{\Delta I_s}{\Delta t}$$

using  $M$  calculated in #11.

11) What is the mutual inductance between the solenoid and the short coil?

The mutual inductance of a coil is given by

$$M = \frac{\phi_2}{I_1}$$

Since the solenoid is enclosed by the outer loop, the flux through the outer loop will be the same as the flux through the solenoid. Flux is given by

$$\phi = \int B \cdot dA = BA$$

Basically this equation will be true as long as the magnetic field does not vary over the area of the object. It may vary with time just as long as it is uniform at any given instant. The total flux in the coil is the number of turns in the coil multiplied by the flux in each turn. Thus,

$$\begin{aligned}
 M &= \frac{N_c \phi_2}{I_1} = \frac{N_c B_1 A_1}{I_1} = \frac{N_c \mu_0 n I_1 A_1}{I_1} \\
 &= N_c \mu_0 n A_1 = N_c \mu_0 n (\pi r_1^2)
 \end{aligned}$$

Remember to convert  $n$  from turns/cm to turns/m.

12) Now reverse the situation. If the current in the short coil is ramped up steadily from zero to  $I_s = 2.90 \text{ A}$  over a time interval of  $25.0 \text{ ms}$ , what is the magnitude of the emf in the solenoid while the current in the coil is changing?

$I_c = \text{Given}$   
 $t = \text{Given}$

Realizing that the mutual inductance  $M$  is a constant for a given configuration of wire, the EMF is simply given by the equation

$$E_2 = -M \frac{dI_c}{dt}$$

Which we can calculate by

$$E_2 = -M \frac{\Delta I_c}{\Delta t}$$

13) A very long solenoid with a circular cross section and radius  $r = 5.10 \text{ cm}$  with  $n = 1.60 \times 10^4 \text{ turns/m}$  has a magnetic energy density  $u_B = 7.80 \text{ mJ/m}^3$ . What is the current in the solenoid?

$r = \text{Given}$   
 $n = \text{Given}$   
 $u_B = \text{Given}$

The equation for magnetic energy density is

$$u_B = \frac{B^2}{2\mu_0}$$

which we can use to solve for  $B$ :

$$\sqrt{2\mu_0 u_B} = B$$

Knowing the magnetic field of a solenoid, we can now find the current:

$$\begin{aligned} \sqrt{2\mu_0 u_B} &= B = \mu_0 n I \\ I &= \frac{\sqrt{2\mu_0 u_B}}{\mu_0 n} \end{aligned}$$

14) What is the total energy stored in the solenoid if its length is  $0.780 \text{ m}$ ? (Neglect end effects.)

$l = \text{Given}$

The easiest way to do this problem is to realize the magnetic energy density given in #13 is just the magnetic field energy per volume. Thus, if we can find the volume of a solenoid of length  $l$ , we can find the energy stored in it. A solenoid is basically a cylinder. The volume of which is

$$V = \pi r^2 l$$

Thus, the energy is

$$U = u_B V = u_B \pi r^2 l$$

15) The switch in the following circuit has been open for a long time. The switch is then closed. The current through the battery immediately after the switch is closed is  $I(t = 0)$  and the value at very large  $t$  is  $I(t = \infty)$ . Calculate the ratio  $I(t = 0)/I(t = \infty)$ .

$E = \text{Given}$   
 $R_1 = \text{Given}$   
 $R_2 = \text{Given}$   
 $L = \text{Given}$

An inductor has mainly opposite properties than those of a capacitor. Thus, initially it acts like an infinite resistance and after a long time it acts like a piece of wire. The ratio of the current initially ( $I_0$ ) to the final current ( $I_\infty$ ) can be calculated in two steps.

1) Since no current initially flows through the inductor, it must flow through the other two resistors. Imagine the circuit as being open where the inductor is so that the two resistors are in series. Using Ohm's law,

$$I_0 = \frac{V}{R} = \frac{E}{R_1 + R_2}$$

2) After a long time, the inductor acts like a piece of wire (effectively zero resistance). Thus, all the current will flow through it. Imagine  $R_2$  has been removed from the circuit. Then, only  $R_1$  determines the current.

$$I_\infty = \frac{V}{R} = \frac{E}{R_1}$$

Taking the ratio of these two yields the final answer

$$\frac{I_0}{I_\infty} = \frac{\frac{E}{R_1 + R_2}}{\frac{E}{R_1}} = \frac{R_1}{R_1 + R_2}$$

16) A circular coil with one turn is in a perpendicular (time dependent) magnetic field given by  $B = 1.500 - 0.0800t$  *Tesla*, where time  $t$  is in seconds. The induced voltage in the loop is  $2.56$  *V*. Calculate the radius of the coil.

$B(t) = \text{Given}$   
 $E = \text{Given}$

From Faraday's law, we know the induced EMF is given by

$$E = -\frac{d\phi}{dt} = -\frac{d}{dt} \left( \int B \cdot dA \right) = -\frac{d}{dt} BA$$

$$E = -A \frac{dB}{dt}$$

$$A = \pi r^2 = -\frac{E}{\frac{dB}{dt}}$$

$$r = \sqrt{-\frac{E}{\pi \frac{dB}{dt}}}$$

The signs work out okay since the derivative of  $B(t)$  is negative.

17) A 230 turn conducting coil with a radius of 15.7 *cm* rotates at a frequency of  $f = 70.0 \text{ Hz}$  in a magnetic field  $B = 0.210 \text{ T}$ . Calculate the generated *rms emf*. The rms (root-mean-square) value of a sinusoidal quantity is the amplitude divided by root of 2.

$N = \text{Given}$

$r = \text{Given}$

$f = \text{Given}$

$B = \text{Given}$

Example 31-6 on page 792 of the text is exactly the same as this problem. The spinning coil is a generator. The equation for the induced EMF as a function of time is derived. However, we are only concerned with the peak voltage. The equation given is

$$E = 2\pi N f B A \sin(2\pi f t)$$

The EMF will be maximum when sin is 1, so we can ignore the last term. Simply plugging into the equation (where  $A$  is the area of the coil)

$$E = 2\pi N f B \pi r^2 = 2\pi^2 N f B r^2$$

The rms EMF is the maximum voltage divided by the root of 2. Thus,

$$E_{rms} = \frac{2\pi^2 N f B r^2}{\sqrt{2}} = \sqrt{2}\pi^2 N f B r^2$$