Solution Derivations for Capa #12

1) A hoop of radius 0.200 m and mass 0.460 kg, is suspended by a point on it's perimeter as shown in the figure. If the hoop is allowed to oscillate side to side as a pendulum, what is the period of small oscillations?

$R =$ Given
$M =$ Given

Since the hoop is pivoting about a point that is not the center of mass, the rotational inertia about its pivot point must be calculated. Use the parallel axis theorem.

$$I = I_{CM} + MR^2$$
$$I = MR^2 + MR^2 = 2MR^2$$

As this is a physical pendulum, the formula for period is $T = 2\pi\sqrt{\frac{I}{MgL}}$ where L is the radius. So,

$$T = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}$$

Note that the masses cancel and you only need the radius to answer the question. Period has units of seconds.

2) The diagram shows a simple pendulum consisting of a mass M suspended by a thin string. The mass swings back and forth between θ_0. The mass M is 1.51 kg and the length of the pendulum is 89.0 cm. If $\theta_0 = 49.9°$, calculate the kinetic energy of the mass when $\theta = 32.3°$.

$m =$ Given
$L =$ Given
$\theta_0 =$ Given
$\theta =$ Given

The easiest way to solve this problem is to find the total energy of the system. Since it will never change, the kinetic energy can then be calculated at any point. The pendulum is at rest at its highest point (angle $= \theta_0$). The potential energy will be equal to the total energy.

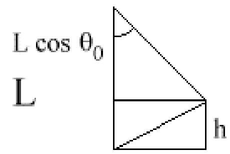
$$E_{tot} = PE = mgh$$

To find the height, look at the diagram and it becomes apparent.

$$h = L\left(1 - \cos\theta_0\right)$$

So,

$$E_{tot} = mgL\left(1 - \cos\theta_0\right)$$



When the angle $= \theta$, then you can calculate the potential energy at the point using the formula above ($PE = mgL\,(1 - \cos\theta)$). Remember that $KE + PE = E_{tot}$, so you can find the kinetic energy by

$$\begin{aligned} KE &= E_{tot} - PE \\ KE &= mgL\,(1 - \cos\theta_0) - mgL\,(1 - \cos\theta) = mgL\,[(1 - \cos\theta_0) - (1 - \cos\theta)] \\ &= mgL\,(1 - 1 - \cos\theta_0 + \cos\theta) = mgL(\cos\theta - \cos\theta_0) \end{aligned}$$

Note: Your length may be given in centimeters; remember to convert to meters. Also, the angle is probably specified in degrees, so don't switch your calculator to radians yet.

3) A 29.0 kg block at rest on a horizontal frictionless air track is connected to the wall via a spring. The equilibrium position of the mass is defined to be at $x = 0$. Somebody pushes the mass to the position $x = 0.350\ m$, then lets go. The mass undergoes simple harmonic motion with a period of 5.60 s. What is the position of the mass 4.480 s after the mass is released?

m = Given
x_0 = Given (displacement from equilibrium)
T = Given
t = Given
x = position at time t = ?

This problem involves using the formula for simple harmonic motion. The mass is not used at all for the solution. To find the position at the given time, use the formula

$$x = A\cos(\omega t + \phi)$$

A is the amplitude, that is, the displacement from equilibrium or what is called x_0 here.

$$\omega = \frac{2\pi}{T}$$

and ϕ is zero in this case because the mass is initially at the greatest displacement. The reason being is that the mass is brought to an initial displacement and let go. So the equation becomes,

$$x = x_0 \cos\left(\frac{2\pi}{T} t\right)$$

Remember to set your calculator in radians for this problem.

4) Consider the same mass and spring discussed in the previous problem. What is the magnitude of the maximum acceleration the mass undergoes during its acceleration?

The acceleration is simply the second derivative of the position function. Since the position function was calculated in the previous problem, simply differentiate it twice.

$$\begin{aligned} x' &= v = -\frac{2\pi}{T}x_0 \sin\left(\frac{2\pi}{T}t\right) \\ x'' &= v' = a = -\frac{4\pi^2}{T^2}x_0 \cos\left(\frac{2\pi}{T}t\right) \end{aligned}$$

CAPA is looking for the magnitude of the largest acceleration the mass. Thus, it can be shown that the cosine function has a maximum value when its argument is 0. So,

$$|a| = \frac{4\pi^2}{T^2}x_0 \cos 0 = \frac{4\pi^2}{T^2}x_0$$

Make sure to keep your calculator in radians.

5) A 37.0 kg block at rest on a horizontal frictionless table is connected to the wall via a spring with a spring constant $k = 19.0\ N/m$. A $3.10 \times 10^{-2}\ kg$ bullet travelling with a speed of 500 m/s embeds itself in the block. What is the amplitude of the resulting simple harmonic motion? Recall that the amplitude is the maximum displacement from equilibrium.

m_1 = Given (mass of block)
k = Given
m_2 = Given (mass of bullet)
v_2 = Given

We can start analyzing this situation by using conservation of momentum for the bullet-block system. Since this is an inelastic collision, only momentum is conserved. The equation is

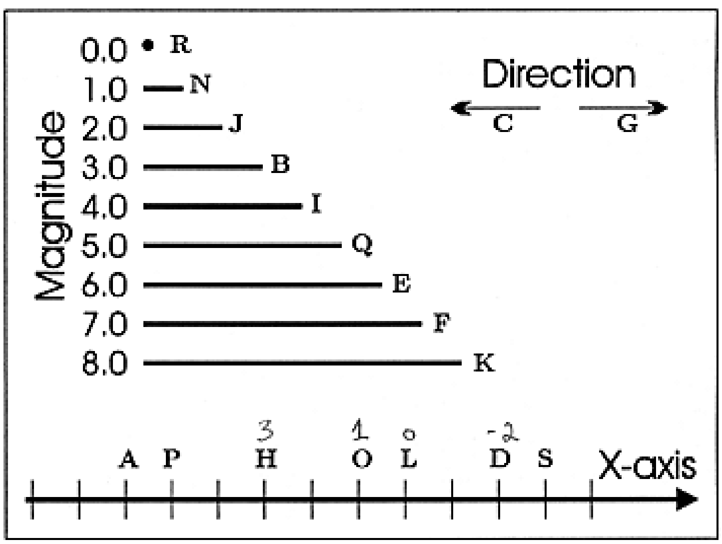
$$\begin{aligned} m_1v_1 + m_2v_2 &= (m_1 + m_2)\, v_f \\ 0 + m_2v_2 &= (m_1 + m_2)v_f \\ v_f &= \frac{m_2v_2}{m_1 + m_2} \end{aligned}$$

However, after the bullet embeds itself into the block, the energy converted into heat will have lower the total energy the bullet-block system has to convert to potential energy. Since the bullet-block system will have an initial velocity, it will have no initial potential energy and it will just start compressing the spring. This kinetic energy will be transformed into the potential energy of the spring. Thus, to relate them,

$$KE_i + PE_i = KE_f + PE_f$$

$$\begin{aligned}
\frac{1}{2}(m_1+m_2)v_f^2+0 &= 0+\frac{1}{2}kx^2 \\
\frac{1}{2}(m_1+m_2)\left(\frac{m_2v_2}{m_1+m_2}\right)^2 &= \frac{1}{2}kx^2 \\
\frac{m_2^2v_2^2}{m_1+m_2} &= kx^2 \\
x &= \sqrt{\frac{m_2^2v_2^2}{k(m_1+m_2)}} = \frac{m_2v_2}{\sqrt{k(m_1+m_2)}}
\end{aligned}$$

6) A mass, connected on either side by springs which obey Hooke's Law, oscillates along the x-axis with no friction. When the mass is at **O** the force acting on the mass is represented by (**N**, **G**), and when it is at **H**, it is represented by (**B**, **G**). Where the first letter labels the magnitude of the force in N and the second letter gives the direction. The maximum value of the x-acceleration occurs at **A**. What force acts on the mass when it is at **D**? Enter you answer as, for example: **K**, **G**



This one is very difficult to explain, but is a very simple problem. The sentence "When the mass is at A the force acting on the mass is represented by (B, C), and when it is at D, it is represented by (E, F)" has all the important information. Obviously, the variables will be different on your CAPA sheet. What's important to figure out is what the scale on the x-axis represents. For this purpose, say the right direction is positive and the left direction is negative. Then, label the point A on your graph as either positive or negative magnitude B. If C points to the right, then B is positive. If C points to the left, then B is negative. Next, label point D as either positive or negative magnitude E. If F points right, E is positive. If F points left, E is negative. Find the distance between A and D, that is, calculate E-B. Then divide by how many tick marks there are on the graph between points E and B. That is the value that each tick represents (it will probably be 1). You can then find a tick mark marking zero, and also the mark representing the location you are asked to find. I have included my problem as an example of how to do this.

7) What is the location when the mass has the maximum speed? The format of your answer is, for example, R. (There are two locations where the mass has zero speed, but only one of the locations is in your graph.)

The mass has maximum speed where the letter of your origin is located (when the magnitude is zero). (L on my problem)

8) At what location is the speed zero?

The speed is zero where the acceleration has a maximum value. This maximum value was probably given to you in part 6. (A on my problem)

9) A transverse mechanical wave is traveling along a string lying along the x-axis. The displacement of the string as a function of position and time, $y(x,t)$, is described by the following equation:

$$y(x,t) = 0.0220 * \sin(4.80x - 152t)$$

where x and y are in *meters* and the time is in *seconds*. What is the wavelength of the wave?

The equation of a wave is given by

$$y(x,t) = A\sin(kx - \omega t)$$

where constants have been filled in for A, k, and ω. Remember that k is defined to be $\frac{2\pi}{\lambda}$, so $\lambda = \frac{2\pi}{k}$. Wavelength is measured in meters when x and y are in meters.

10) What is the velocity of the wave? (Define positive velocity along the positive x-axis.)

To find the velocity, remember that $v = \lambda f = \frac{\lambda}{T}$. But there is an easier way. Velocity is also equal to the quotient of ω and k. That is, $v = \frac{\omega}{k}$.

11) What is the maximum speed in the y-direction of any piece of the string. (Give a positive answer for speed.)

The speed in the y direction of any piece of the string is simply the partial derivative of the displacement function with respect to time. That is,

$$v = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

Since the question is asking about the maximum speed, find the maximum of that function. Clearly, it will have a maximum when the cosine factor is equal to 1. So the maximum speed is

$$|v| = \omega A$$

12) A painter wishes to know whether or not he can safely stand on a ladder. The ladder has a mass $M_1 = 12\ kg$ which is uniformly distributed throughout its length $L = 7.1\ m$. The ladder is propped up at an angle $\theta = 60°$. The coefficient of static friction between the ground and the ladder is $\mu_s = 0.31$, and the wall against which the ladder is resting is frictionless. Calculate the maximum mass of the painter for which the ladder will remain stable when he climbs a distance $d = 5.5\ m$ up the ladder. (The painter's mass might be so low that only Lilliputian painters can safely ascend the ladder.)

m_1 = Given (mass of ladder)
L = Given
θ = Given
μ_s = Given
d = Given
m_2 = ? (mass of painter)

Since the ladder is in equilibrium, the sum of the forces and torques must be zero. The torque of the ladder will be positive as will the torque of the painter. The torque exerted on the ladder by the wall will be negative (use the right-hand rule). The normal force and frictional force exert no torque since the pivot point is taken to be the base of the ladder. Summing the forces and torques,

$$\begin{aligned}\sum F_x &= F_{wall} - \mu_s N = 0 \\ \sum F_y &= N - m_2 g - m_1 g \\ \sum \tau &= \tau_{ladder} + \tau_{man} - \tau_{wall} = 0\end{aligned}$$

So we know that

$$F_{wall} = \mu_s N \tag{1}$$

$$N = m_1 g + m_2 g \tag{2}$$

$$\tau_{ladder} + \tau_{man} = \tau_{wall} \tag{3}$$

Equation (2) can be substituted back into equation (1).

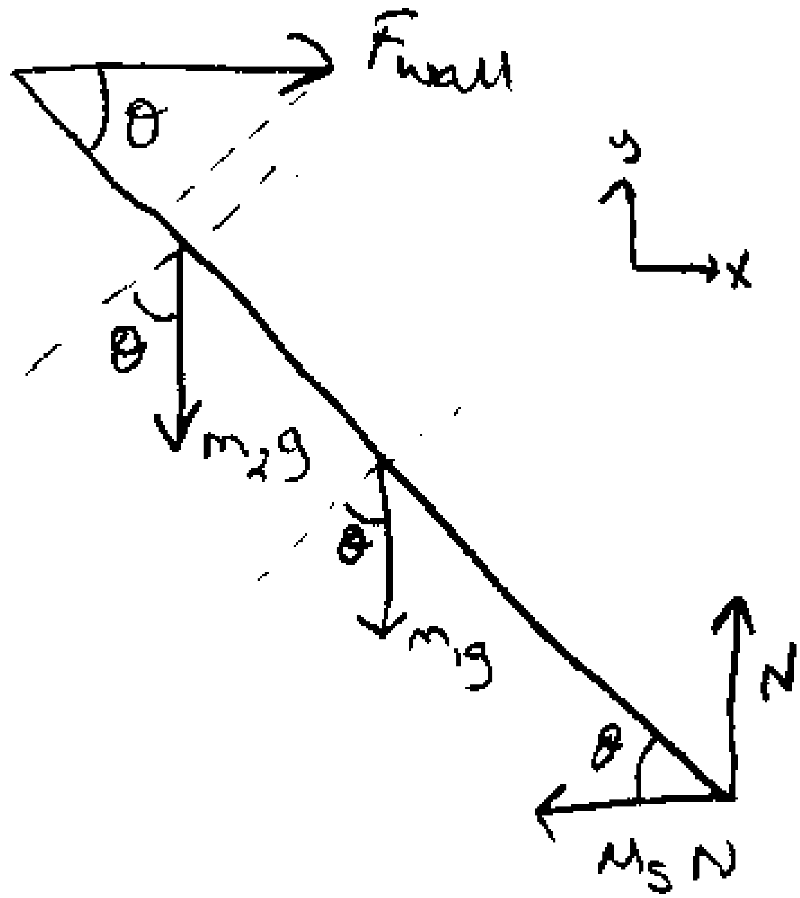
$$F_{wall} = \mu_s (m_1 g + m_2 g)$$

Torque can be thought of as the lever arm acting perpendicular to the ladder times the length to the point where the force is applied. So equation (3) becomes

$$\frac{L}{2} m_1 g \cos\theta + d * m_2 g \cos\theta = L * F_{wall} \sin\theta$$

Substituting the value of F_{wall}, the equation becomes

$$\frac{L}{2} m_1 g \cos\theta + d * m_2 g \cos\theta = L * \mu_s (m_1 g + m_2 g) \sin\theta$$



Solving for m_2,

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\begin{aligned}
\frac{L}{2} m_1 g \cos\theta + d * m_2 g \cos\theta &= L * \mu_s m_1 g \sin\theta + L * \mu_s m_2 g \sin\theta \\
\frac{L}{2} m_1 g \cos\theta - L * \mu_s m_1 g \sin\theta &= L * \mu_s m_2 g \sin\theta - d * m_2 g \cos\theta \\
\frac{L}{2} m_1 g \cos\theta - L * \mu_s m_1 g \sin\theta &= m_2 \left(L * \mu_s g \sin\theta - d * g \cos\theta\right) \\
\frac{\frac{L}{2} m_1 g \cos\theta - L * \mu_s m_1 g \sin\theta}{\left(L * \mu_s g \sin\theta - d * g \cos\theta\right)} &= m_2 \\
\frac{\frac{L}{2} m_1 \cos\theta - L * \mu_s m_1 \sin\theta}{\left(L * \mu_s \sin\theta - d * \cos\theta\right)} &= m_2
\end{aligned}
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(The g's cancel). Make sure to put your calculator back into degrees.